SOME PROBLEMS OF UNSTEADY HEAT CONDUCTION THEORY FOR A TWO-LAYER SEMI-SPACE

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We shall examine the unsteady temperature distribution in a two-layer semi-space, at point (x_0, y_0) of which a concentrated, impulsive heat source is located, the boundary being thermally insulated.

The problem in question reduces to the system of equations

$$\frac{\partial^2 T_i}{\partial x^2}+\frac{\partial^2 T_i}{\partial y^2}=\alpha_i\frac{\partial T_i}{\partial t}-\frac{Q}{k_i}\delta(x-x_0,\ y-y_0,\ t)\quad (i=1,\ 2),$$

$$\left.\begin{array}{l}-\infty<x\leqslant 0\ (i=1)\\ 0\leqslant x<\infty\quad (i=2)\end{array}\right\},\ 0\leqslant y<\infty, \tag{1}$$

the solution of which must satisfy the initial condition

$$T_i|_{t=0}=0, \tag{2}$$

the edge condition

$$\left.\frac{\partial T_i}{\partial y}\right|_{y=0}=0, \tag{3}$$

as well as the following requirements at the medium interface:

$$T_1\Big|_{x=0}=T_2\Big|_{x=0},\quad k_1\frac{\partial T_1}{\partial x}\Big|_{x=0}=k_2\frac{\partial T_2}{\partial x}\Big|_{x=0}. \tag{4}$$

To obtain an exact solution of this problem, it is convenient to apply the method of integral transforms (a Laplace transformation with respect to the variable t, and a cosine and sine Fourier transformation with respect to the coordinates y and x, respectively). After some operations, the general solution of the problem may be obtained in the following form:

$$T_1(x,\ y,\ t)=$$

$$=\frac{2Q}{\pi}\int_0^\infty\cos\lambda y_0\cos\lambda y\,d\lambda\,\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\exp(pt+\beta_1 x-\beta_2 x_0)}{k_1\beta_1+k_2\beta_2}\,dp, \tag{5}$$

$$T_2(x,\ y,\ t)=$$

$$=\frac{2Q}{\pi}\int_0^\infty\cos\lambda y_0\cos\lambda y\,d\lambda\,\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left\{\frac{\exp(pt-\beta_2 x-\beta_2 x_0)}{k_1\beta_1+k_2\beta_2}+\right.$$

$$\left.+\frac{1}{2k_2\beta_2}\left(\exp[pt-\beta_2|x_0-x|]-\exp[pt-\beta_2(x_0-x)]\right)\right\}dp, \tag{6}$$

where

$$\beta_i^2=\lambda^2+\alpha_i p,\ \operatorname{Re}\beta_i>0\quad (i=1,\ 2). \tag{7}$$

Without carrying out the extensive calculations to carry the solution to simple quadratures, we shall turn to the particular case, when there is a source at the coordinate origin, and find the temperature variation law at the medium interface.

Putting $x = x_0 = y_0 = 0$ in (5) and (6), and carrying out the appropriate transformations, we obtain

$$T_0=T(0,\ y,\ t)=$$

$$=\frac{Q}{2\pi k_1 t}\exp\left(-\frac{\alpha_1 y^2}{4t}\right)\frac{2}{\gamma^2-1}\left\{\gamma\exp\left[\frac{(\nu-1)\alpha_1 y^2}{4t\nu}\right]-1+\right.$$

$$+\sqrt{\pi}\gamma\omega\frac{\sqrt{\alpha_1}y}{2\sqrt{t}}\left[\Phi\left(\omega\frac{\sqrt{\alpha_1}y}{2\sqrt{t}}\right)-\right.$$

$$\left.\left.-\Phi\left(\gamma\omega\frac{\sqrt{\alpha_1}y}{2\sqrt{t}}\right)\right]\exp\left(\gamma^2\omega^2\frac{\alpha_1 y^2}{4t}\right)\right\}, \tag{8}$$

where

$$\gamma=k_2/k_1,\quad \nu=\alpha_1/\alpha_2, \tag{9}$$

and the notation

$$\omega^2=\frac{\nu-1}{\nu(\gamma^2-1)},\qquad \Phi(z)=\frac{2}{\sqrt{\pi}}\int_0^z\exp(-t^2)\,dt \tag{10}$$

has been introduced.

For $\gamma = 1$ ($k_1 = k_2 = k$) formula (8) has the form

$$T_0=\frac{Q}{2\pi kt}\exp\left(-\frac{\alpha_1 y^2}{4t}\right)\frac{\nu}{\nu-1}\frac{4t}{\alpha_1 y^2}\left[\exp\left(\frac{\nu-1}{\nu}\frac{\alpha_1 y^2}{4t}\right)-1\right], \tag{11}$$

whence, for a homogeneous medium with parameters k_1 and α_1($\gamma = \nu = 1$), the well-known formula

$$T_0^I=\frac{Q}{2\pi k_1 t}\exp\left(-\frac{\alpha_1 y^2}{4t}\right) \tag{12}$$

is obtained.

Simple intuitive results describing the thermal process examined may be obtained by comparing the values of T_0 and T_0^1. We shall

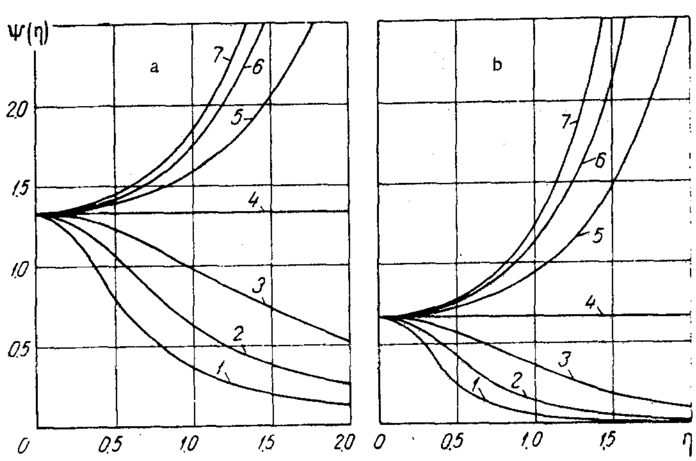

Fig. 1. Graphs of reduced medium interface temperature with a) $\gamma = 0.5$ and b) 2, and the parameter ν equal to 1) 0.125; 2) 0.25; 3) 0.5; 4) 1; 5) 2; 6) 4; 7) 8.

introduce the function $\Psi(\eta)$ to describe their ratio:

$$\Psi(\eta) = T_0/T_0^1, \quad \eta = \sqrt{\alpha_1}\, y/2\sqrt{t}. \tag{13}$$

Examination shows that when $\eta = 0$

$$\Psi(0) = 2/(1+\gamma), \tag{14}$$

i.e., it is independent of ν, and when $\eta \to \infty$ the behavior of $\Psi(\eta)$ is determined by the asymptotic expression

$$\Psi(\eta) \underset{\eta\to\infty}{\approx} \frac{\nu}{\gamma^2(\nu-1)}\left[\gamma^3 \exp\left(\frac{\nu-1}{\nu}\eta^2\right)-1\right] \tag{15}$$

$$(\gamma, \ \nu \neq 1).$$

Thus, for a given value of the parameter γ and various values of the parameter ν, the function $\Psi(\eta)$ changes from the same value $2/(1+$ $+\gamma)$, but when $\eta \to \infty$ it behaves in a substantially different way, depending on the values of the parameter ν; namely, when $\nu > 1$ it grows without bound, while when $\nu < 1$ it tends to zero (when $\nu = 1$ $\Psi(\eta) \equiv \Psi(0)$).

The figure presents graphs of the function $\Psi(\eta)$, drawn on the basis of calculations according to formulas (8)-(13) carried out on a "Minsk-2" electronic computer.

NOTATION

T—temperature; t—time; x, y—rectangular coordinates; Q—volume density of heat source; α—reciprocal of thermal diffusivity; k—thermal conductivity; δ—delta function.

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